THE APPROXIMATE SOLUTION OF THE EQUATION FOR THE SUPERSONIC FLOW OF A GAS

P. M. Kolesnikov UDC 533.6

A new approximation of the Chaplygin function is proposed in this paper, one that is suitable at $1 < M < 3$, making it possible to find analytical solutions for the equations of supersonic gas flow. An explicit form of the Riemann function has been derived for the potential equation and the stream function, and the limit transitions to the Tricomi equation or to the Euler–Poisson–Darboux equation are examined.

1. Let us examine the integration of a canonical system of equations for the plane potential supersonic steady-state motion of a gas at velocities in excess of the speed of sound [1, 2]:

$$\frac{\partial^2\varphi}{\partial\xi\partial\eta} = \frac{1}{4}\frac{d}{dt}\ln H\left(\frac{\partial\varphi}{\partial\xi}+\frac{\partial\varphi}{\partial\eta}\right)=0,$$
$$\frac{\partial^2\psi}{\partial\xi\partial\eta} = -\frac{1}{4}\frac{d}{dt}\ln H\left(\frac{\partial\psi}{\partial\xi}+\frac{\partial\psi}{\partial\eta}\right), \tag{1}$$

where ξ and η are characteristic variables; $t = \xi + \eta$; φ is the velocity potential; ψ is the stream function; H is the Chaplygin function. For a specified Chaplygin function $H = H(t)$ the gas density ρ and the gas velocity v are determined from the equations

$$\frac{d\rho}{dt} = -\frac{\rho^2 H + 1}{\sqrt{H}}, \tag{2}$$

$$\frac{dv}{dt} = \frac{v}{\rho\sqrt{H}}. \tag{3}$$

We know that because of the complex function $H = H(t)$ Eq. (1) is not easily solved; a method has therefore been developed for the integration of these equations, this method based on replacement of the variable coefficient in (1) by an approximate coefficient for which we can construct a solution for (1) and find all of the parameters of the gas. The accuracy of the approximate calculations depends on the correct choice of this coefficient. For transonic velocities, it is the Tricomi approximation that is most common [1], i.e.,

$$\frac{1}{4}\frac{d}{dt}\ln H = \frac{1}{6t}, \tag{4}$$

and the Tricomi equations here represent a special case of the Euler–Poisson–Darboux equation with fractional coefficients. The solutions for (1) in this case are expressed in terms of hypergeometric functions. However, Eq. (4) with the real function H is satisfied only for small t ($0 < t < 0.05$), which corresponds to Mach numbers of $1 < M < 1.2$. The Tricomi approximation is not suitable for greater values of the Mach numbers. Khristianovich [3] proposed the following approximation:

$$\frac{1}{4}\frac{d}{dt}\ln H = \frac{1}{a+t}, \tag{5}$$

whose advantage lies in the fact that (1) are solved in elementary functions which allow us simply to solve the boundary-value problems. However, with small t, i.e., in the effective region of approximation (4)

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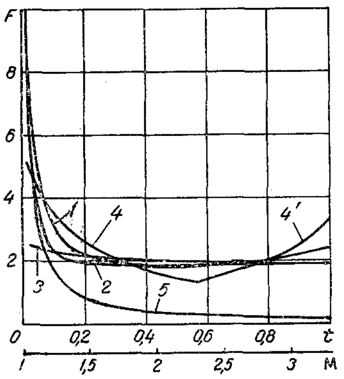

Fig. 1

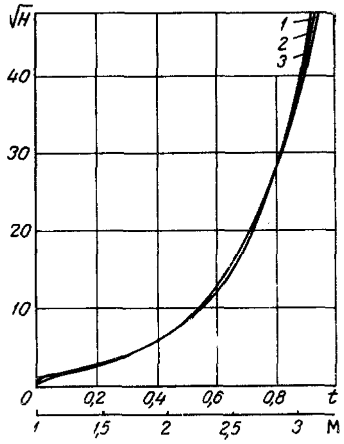

Fig. 2

Fig. 1. Comparison of the true function $F(t) = (1/4)(d/dt)\ln H$ (curve 1) with the various approximations: 2) $F(t) = 1.9\coth 1.9 \cdot 6t$; 3) $F(t) = 1.9\coth 1.9(t + 0.5)$; 4) Khristianovich approximation $F(t) = 1/(t + 0.185)$; 4') Kristianovich approximation $F(t) = 1/(1.3 - t)$; 5) Tricomi approximation $F(t) = 1/6t$.

Fig. 2. Comparison of the true function $\sqrt{H}$ (curve 1) with the approximations: $\sqrt{H} = c^2 \mathrm{sh}^{2n/m}\, m(t + a)$, $c^2 = 1.65$; $n = 1.9$; $m = 6n$, $a = 0$; 3) $\sqrt{H} = c^2 \mathrm{sh}^2 m(t + a)$, $c^2 = 0.825$; $m = 1.9$; $a = 0.5$.

and, on the other hand, for $M > 1.8$, the values of the coefficient in (1), calculated from (5), differ substantially from the actual values. Another series of approximations [4-7] has been proposed for low supersonic velocities.

In this paper we examine a new approximation which in the case of transonic flows goes over into the Tricomi approximation (4), but is suitable with sufficient accuracy at $1 < M < 3$.

2. Let us choose the coefficient in (1) in the form

$$\frac{1}{4}\,\frac{d}{dt}\,\ln H = n\,\mathrm{cth}\, m\,(t + a), \tag{6}$$

where n, m, and a are arbitrary constants. With low values for t the right-hand member of (6) can be expanded in series

$$n\,\mathrm{cth}\, m\,(t + a) = n\left[\frac{1}{m\,(t + a)} + \frac{m\,(t + a)}{3} - \frac{m^3\,(t + a)^3}{45} + \cdots\right], \tag{7}$$

from which we can see that for low values of t Eq. (6) changes into the Tricomi approximation (4), if we choose $m = 6n$ and $a = 0$. The constants in (6) can be determined from the condition of coincidence between (6) and the real function at any points whatsoever. Figure 1 shows a comparison of the various approximations with the true curve, and from this we can see that approximation (6) more accurately reflects the shape of the true curve than do (4) and (5), and over a greater interval.

Relationship (6) can be regarded as an equation for the determination of H on the basis of the specified right-hand member. Integration yields

$$H = c^4\,\mathrm{sh}^{\frac{4n}{m}}\, m\,(t + a), \tag{8}$$

where c^4 is the integration constant which can be selected from the condition that the real function H coincides with approximation (8). The real function H is compared with the approximations in Fig. 2.

3. Let us examine the integration of (1) in the case of approximation (6). In this case they assume the form

$$\frac{\partial^2\varphi}{\partial x\partial y} = \frac{n}{m}\,\mathrm{cth}\,\sigma\left(\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}\right), \tag{9}$$

$$\frac{\partial^2\psi}{\partial x\partial y} = -\frac{n}{m}\operatorname{cth}\sigma\left(\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}\right), \tag{10}$$

where $m(\xi + a) = x$, $m\eta = y$, $\sigma = x + y$. The solution of (9) and (10) will be sought by the Riemann method. For example, let us take Eq. (10). The associated equation, defining the Riemann function $R(x, y, x_0, y_0)$, will be

$$\frac{\partial^2 R}{\partial x\partial y} - \frac{n}{m}\operatorname{cth}\sigma\left(\frac{\partial R}{\partial x}+\frac{\partial R}{\partial y}\right) + \frac{2nR}{m\operatorname{sh}^2\sigma} = 0. \tag{11}$$

We will seek solution (11) in the form of

$$R(x, y, x_0, y_0) = V\exp\left(\frac{n}{m}\int_{\sigma_0}^{\sigma}\operatorname{cth}\sigma' d\sigma'\right), \tag{12}$$

where $V(x, y, x_0, y_0)$ is the new unknown function. For its determination we derive the following equation on substitution of (12) into (11):

$$\frac{\partial^2 V}{\partial x\partial y} - \left(\frac{n}{m}\right)^2 V + \left(\frac{n}{m} - \frac{n^2}{m^2}\right)\frac{V}{\operatorname{sh}^2\sigma} = 0. \tag{13}$$

For the integration of (13), let us examine two auxiliary equations

$$\frac{\partial^2 V_1}{\partial x\partial y} - \left(\frac{n}{m}\right)^2 V_1 = 0, \tag{14}$$

$$\frac{\partial^2 V_2}{\partial x\partial y} + \left(\frac{n}{m} - \frac{n^2}{m^2}\right)\frac{V_2}{\operatorname{sh}^2\sigma} = 0. \tag{15}$$

Solution (14) is expressed in terms of a zeroth-order Bessel function of imaginary argument

$$V_1(x, y, x_0, y_0) = I_0\left(2\frac{n}{m}\sqrt{(x-x_0)(y-y_0)}\right). \tag{16}$$

Equation (15) can be turned into a hypergeometric equation, if we seek the solution for the function V_2 in the form

$$V_2(x, y, x_0, y_0) = G\left[\tau(x, y, x_0, y_0)\right], \tag{17}$$

$$\tau = -\frac{\operatorname{sh}(x-x_0)\operatorname{sh}(y-y_0)}{\operatorname{sh}(x_0+y_0)\operatorname{sh}(x+y)}. \tag{18}$$

Substituting (17) and (18) into (15), we find the hypergeometric equation

$$\tau(1-\tau)\frac{d^2G}{d\tau^2} + (1-2\tau)\frac{dG}{d\tau} + \left(\frac{n}{m} - \frac{n^2}{m^2}\right)G = 0. \tag{19}$$

The hypergeometric function F serves as a solution of (19):

$$V_2 = F\left(a, b, 1, -\frac{\operatorname{sh}(x-x_0)\operatorname{sh}(y-y_0)}{\operatorname{sh}(x_0+y_0)\operatorname{sh}(x+y)}\right),$$
$$a = \frac{1}{2} \mp \sqrt{\frac{1}{4} + \left(\frac{n}{m} - \frac{n^2}{m^2}\right)}, \quad b = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \left(\frac{n}{m} - \frac{n^2}{m^2}\right)}. \tag{20}$$

Olevskii [8] demonstrated that with solutions (16) and (20) for the auxiliary equations (14) and (15) it is possible to construct the Riemann function of Eq. (11) in the form

$$R(x, y, x_0, y_0) = \left(\frac{\operatorname{sh}\sigma}{\operatorname{sh}\sigma_0}\right)^{\frac{n}{m}}\left\{I_0\left(2\frac{n}{m}\sqrt{(x-x_0)(y-y_0)}\right)\right.$$
$$\left. + \int_{y-y_0}^{x-x_0} I_0\left(2\frac{n}{m}\sqrt{(x-x_0)z}\right)\frac{\partial}{\partial z}F\left(a, b, 1, -\frac{\operatorname{sh}z\operatorname{sh}(y-y_0)}{\operatorname{sh}y_0\operatorname{sh}(z+y)}\right)dz\right\}. \tag{21}$$

The Riemann function of (9) can be found in analogous fashion:

$$R(x, y, x_0, y_0) = \left(\frac{\operatorname{sh}\sigma}{\operatorname{sh}\sigma_0}\right)^{-\frac{n}{m}} \left\{ I_0\left(\frac{2n}{m}\sqrt{(x-x_0)(y-y_0)}\right)\right.$$

$$\left. + \int\limits_{y-y_0}^{x-x_0} I_0\left(\frac{2n}{m}\sqrt{(x-x_0)z}\right)\frac{\partial}{\partial z} F\left(a_1, b_1, 1, -\frac{\operatorname{sh} z \operatorname{sh}(y-y_0)}{\operatorname{sh} y_0 \operatorname{sh}(z+y)}\right) dz \right\},$$

$$a_1 = \frac{1}{2} \mp \sqrt{\frac{1}{4} - \left(\frac{n}{m} + \frac{n^2}{m^2}\right)}, \quad b_1 = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \left(\frac{n}{m} + \frac{n^2}{m^2}\right)}. \tag{22}$$

Using the Riemann functions (21) and (22), we can use standard notation to write the gasdynamic boundary-value problems of the Cauchy or Goursa type in integral form [9, 10].

4. Solution of (9) and (10), expressed in terms of the Riemann function (21) and (22), are rather cumbersome, although they provide for the required asymptotic values corresponding to the solutions in the case of the Tricomi approximations, if we choose the following values for the parameters in approximation (6): m = 6n and a = 0. Indeed, in this case expansion (7) for small values of t changes into the Tricomi approximation (4). For small values of t the magnitude of the parameter n is insignificant. Its value must be chosen on the basis of the condition that approximation (6) coincide with the true curve. The calculations show that satisfactory approximations are obtained at values of n = 1.9-2.0. It turns out that varying the parameters m, n, and a enables us to obtain solutions for the equations in simpler form. Let us consider the special case m = n. Function (6) for n = 1.9 and a = 0.5 is shown in Fig. 1, curve 3. We see from Fig. 1 that the approximation is satisfactory in the Mach-number interval 1.2 < M < 3. In this case (13) assumes the form

$$\frac{\partial^2 V}{\partial x \partial y} - V = 0. \tag{23}$$

The Riemann function of (10) is simple in this case, i.e.,

$$R(x, y, x_0, y_0) = \frac{\operatorname{sh}\sigma}{\operatorname{sh}\sigma_0} I_0\left(2\sqrt{(x-x_0)(y-y_0)}\right). \tag{24}$$

Moreover, here we can calculate the parameters ρ and v from (2) and (3). Considering that $H = c^4 \operatorname{sh}^4 m(t + a)$, we note that (2), written in the form

$$\frac{d\rho}{dt} = -\rho^2 c^2 \operatorname{sh}^2 m(t+a) - \frac{1}{c^2 \operatorname{sh}^2 m(t+a)}, \tag{25}$$

is the Riccati equation. With substitution

$$\rho = -\frac{z}{c^2 \operatorname{sh}^2 m(t+a)} \tag{26}$$

we bring (25) to the form

$$\frac{dz}{dt} - 2mz \operatorname{cth} m(t+a) = z^2 + 1. \tag{27}$$

With another substitution

$$z = u - m \operatorname{cth} m(t+a) \tag{28}$$

we bring (27) to the canonical form of the Riccati equation

$$\frac{du}{dt} = u^2 + 1 - m^2 - \frac{2m^2}{\operatorname{sh}^2 m(t+a)}, \tag{29}$$

while (29) with the substitution

$$u = -\frac{1}{w}\frac{dw}{dt} \tag{30}$$

is brought to the second-order equation

$$\frac{d^2 w}{dt^2} = \left(m^2 - 1 + \frac{2m}{\operatorname{sh}^2 m(t+a)}\right) w. \tag{31}$$

The solution of (31) is [11]

$$w = \mathrm{sh}^p\,\vartheta\left(\frac{1}{\mathrm{sh}\,\vartheta}\,\frac{d}{d\vartheta}\right)^p(C_1e^{q\vartheta} + C_2e^{-q\vartheta}), \tag{32}$$

where

$$\vartheta = m(t + a),\ q^2 = \frac{m^2-1}{m},\quad p(p-1) = 2. \tag{33}$$

From (33) we see that p can assume two values: p = 2, p = −1. For specific calculations we can take the solution with one of the roots. Further, with (32) and substitutions (30), (28), and (26), we can find the gas density ρ as a function of t, and knowing $\rho = \rho(t)$, with (3) in quadratures we can determine v = v(t). Because of their cumbersome nature, we will not write these solutions out here.

5. Finally, let us examine one limiting case in which the coefficient in (1) can be assumed to be constant. Indeed, we see from Fig. 1 that this is possible in the Mach-number range 1.5 < M < 3, assuming that

$$\frac{1}{4}\,\frac{d}{dt}\ln H = \mathrm{const} = A. \tag{34}$$

The value of the constant can be set with sufficient accuracy in the interval $1.9 \le A \le 2.0$. This case can be treated as a limiting case of (6) for large values of the argument. Using asymptotic expansions in the case of large values of the argument

$$\underset{x\to\infty}{I_0(x)} \simeq \frac{e^x}{(2\pi x)^{1/2}},\quad \underset{x\to\infty}{\mathrm{sh}\,x} \simeq \frac{e^x}{2}, \tag{35}$$

we can write the Riemann function (24) in the form

$$R(x, y, x_0, y_0) \simeq \frac{\exp\left[(x-x_0) + (y-y_0) + 2\sqrt{A(x-x_0)(y-y_0)}\right]}{2\sqrt{\pi}\left(\sqrt{(x-x_0)(y-y_0)}\right)^{1/2}}. \tag{36}$$

The Riemann function of (9) and (10), in case (34), can be written out directly. For example, let us take Eq. (10)

$$\frac{\partial^2\psi}{\partial x\partial y} = -A_1\left(\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\right),\quad A_1 = \frac{A}{m}, \tag{37}$$

$$R(x, y, x_0, y_0) = \exp\{A_1[(x-x_0)(y-y_0)]\}\, I_0\left(2\sqrt{A_1(x-x_0)(y-y_0)}\right). \tag{38}$$

In (38), taking the asymptotic value of the Bessel function I_0 for large values of the argument, we derive (36).

Let us now consider the small values of the argument in approximation (6). In this case it is more convenient to rewrite the Riemann function (21), i.e.,

$$R(x, y, x_0, y_0) = \left(\frac{\mathrm{sh}\,\sigma}{\mathrm{sh}\,\sigma_0}\right)^{\frac{n}{m}}\left\{F\left(a, b, 1, -\frac{\mathrm{sh}(x-x_0)\,\mathrm{sh}(y-y_0)}{\mathrm{sh}(x_0+y_0)\,\mathrm{sh}(x+y)}\right)\right.$$

$$\left. + \int_{y-y_0}^{x-x_0} F\left(a, b, 1, -\frac{\mathrm{sh}(x-x_0)\,\mathrm{sh}\,z}{\mathrm{sh}\,x_0\,\mathrm{sh}(x+z)}\right)\frac{\partial}{\partial z}\, I_0\left(\frac{2n}{m}\sqrt{z(y-y_0)}\right)dz\right\}. \tag{39}$$

Bearing in mind that for low values of the argument

$$\underset{x\to 0}{I_0(x)} \sim 1,\quad \frac{\partial}{\partial x}\underset{x\to 0}{I_0(x)} = 0, \tag{40}$$

we see that in (39) the second term disappears and the Riemann function is expressed, as in the case of the Tricomi equation, in terms of the hypergeometric function.

NOTATION

φ	is the velocity potential;
ψ	is the stream function;
H	is the Chaplygin function;
ξ and η	are the characteristic variables;
$t = \xi + \eta$;	

ρ is the gas density;
v is the gas velocity;
M is the Mach number;
R is the Riemann function.

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